Asymptotics of the spectrum near the boundaries of spectral clusters for a Hartree-type operator

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We consider the eigenvalue problem for a nonlinear Hartree-type operator in $L^2(\mathbb{R}^2)$:

$$(H_0+\varepsilon\int_{\mathbb{R}^2} W(|q-q'|^2)|\psi(q')|^2\,dq')\psi=\lambda\psi, \qquad \|\psi\|_{L^2(\mathbb{R}^2)}=1, \tag{1}$$

where

$$H_0=-\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial q_1^2}+\frac{\partial^2}{\partial q_2^2}\right)+\frac{q_1^2+q_2^2}{2}$$

is a two-dimensional oscillator, $W(x)=\omega_0+\omega_1 x+\omega_2 x^2$ is an arbitrary polynomial of degree two with real coefficients, $\hbar>0$ and $\varepsilon>0$ are small parameters, and $\varepsilon\ll\hbar$. For definiteness, we consider the case where $\varepsilon=\hbar^2$ and $\omega_2>0$.

A special characteristic of problem (1) is the fact that it belongs to the class of resonance problems, i.e., both frequencies of the two-dimensional oscillator H_0 are equal to 1. But then the ray method and the general theory of Maslov's complex germ cannot be applied. A method for constructing semiclassical asymptotics for equations with frequency resonances was developed by M.V.Karasev in the series of his works starting from [1].

The solutions of equations of the form (1), which correspond to the boundaries of spectral clusters near the eigenvalues of the unperturbed equation (for $\varepsilon=0$) are of perticular interest. In [2], an example of the spectral problem for the two-dimensional perturbed oscillator was used to propose a method for determining series of asymptotic eigenvalues near the boundaries of spectral clusters. This method is based on a new integral representation.

In the present paper, this method is used to obtain asymptotic eigenvalues of problem (1) near the boundaries of spectral clusters. The operator averaging and coherent transformation is applied to problem (1) on the lth irreducible representation of the algebra of symmetries of the unperturbed operator, and the eigenvalue problem in the space P_l of polynomials of degree less than or equal to $l\sim 1/\hbar$ is thus obtained. The desired polynomial satisfies a second-order differential equation. We first study the multiple-point spectral problem in the class of holomorphic functions with zero characteristic exponents at finite singular points. Further, we obtain the asymptotics of the desired polynomial by using the operation of projection on the space P_l, which generalizes the integral Dirac representation.

Near the lower boundaries of spectral clusters, the asymptotic eigenvalues have the form

$$\lambda = \lambda_{k,l} = l\hbar + \hbar + (\omega_0 + 2l\hbar\omega_1 + 6l^2\hbar^2\omega_2)\hbar^2 +$$

$$+(2\omega_1 + 2l\hbar\omega_2(2k+7)))\hbar^3 + O(\hbar^4), \qquad k = 0,1,2,..., \qquad \hbar \to 0 .$$

Here the number $l \in \mathbb{N}$ is of the order of $\hbar^{-1}$ and determines the eigenvalue $\hbar(l+1)$ of the unperturbed operator H_0. The asymptotics near the upper boundaries was studied in [3].

References

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